Closing Today: HW_5C (7.3)
Closing *Thurs*: HW_6A, 6B (7.4, 7.5)

7.4 Partial Fractions

Motivation: We will learn to break-up fractions like:

$$\frac{x^3+4x-4}{x^2(x^2+4)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2+4}$$

Entry Task 1: If I tell you the above relationship is true, then integrate

$$\int \frac{x^3+4x-4}{x^2(x^2+4)}\,dx$$

Entry Task 2: Do you know **long-division**?
Divide in order to fill in the question marks:

$$\frac{576}{11} = ? + \frac{?}{11}$$

Partial Fraction Decomposition

Step 0: Is the fraction *reduced*?

reduced - highest power on top smaller than the highest power on bottom.

If yes, move to step 1.

If not, divide, then move to step 1.

Example:

$$\int \frac{x^2 + x}{x + 3}\, dx$$

Partial Fractions Method Summary

Step 0: Reduce (if needed), see last page.

Step 1: Factor Denominator.
Write out decomposition below:

i) Distinct Linear:

$$\frac{x^2-3}{x(x-1)(x+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+4}$$

ii) Repeated Linear:

$$\frac{5+2x}{(x+3)(x-2)^3} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

iii) Irreducible Quadratic:

$$\frac{4x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

Step 2: Solve for A, B, C ….

Step 3: Integrate

All the integrals in this section look like these:

$$\int \frac{1}{2x+5}\,dx = \frac{1}{2}\ln|2x+5| + C$$

$$\int \frac{1}{(x-4)^2}\,dx = -\frac{1}{x-4} + C$$

$$\int \frac{1}{(x+7)^3}\,dx = -\frac{1}{2}\frac{1}{(x+7)^2} + C$$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9| + C$$

The method uses algebra to rewrite ***any*** rational function as a sum of the integrals like those above.

Example:

$$\int \frac{x+1}{x^2-4}\,dx$$

Example:

$$\int \frac{x+1}{x^3+3x^2}\,dx$$

Example:

$$\int \frac{x^2 - x + 6}{x^3 + 3x} dx$$

Example:

$$\int \frac{x}{x^2 + 4x + 5} dx$$

How to integrate

A. Look for simplifications/substitutions

B. Products/Logs/Inverse Trig → BY PARTS
   Sin/Cos/Tan/Sec combos → TRIG
   Quadratic (under a radical) → TRIG SUB
   Rational Function → PART. FRAC.

C. If nothing seems to work, substitution.
   (u = inside, u = $\sqrt{\ }$, u = trig, u = e^x)

Examples of substitution:

1. $\int e^{\sqrt{x}} dx$

2. $\int \frac{3}{x - 2\sqrt{x}} dx$

3. $\int \frac{\cos(x)}{4 - \sin^2(x)} dx$

4. $\int e^x \cos(e^x) \sin^3(e^x)\, dx$

How would you *start* these?

1. $\int \tan^3(x)\sec(x)\,dx$

2. $\int x^2 \ln(x)\,dx$

3 $\int x\sqrt{5-x^2}dx$

4. $\int \frac{\sqrt{x^2-1}}{x^2}dx$

5. $\int \frac{x^2+1}{x^2-2x-3}dx$

6. $\int x\tan^{-1}(x)dx$

7. $\int \frac{dx}{\sqrt{4x^2+8x-12}}dx$